ELLIPTIC FUNCTIONS AND THETA FUNCTIONS

LECTURE NOTES FOR NOV.22, 24, 2016

Elliptic functions are generalizations of trigonometry functions. While trigonometry functions are periodic functions with one period, the elliptic functions are meromorphic functions on the complex plane with two periods that are not $\mathbb{R}$-colinear.

Historically, elliptic functions were first discovered by Niels Henrik Abel as inverse functions of elliptic integrals, and their theory was improved by Carl Gustav Jacobi; these in turn were studied in connection with the problem of the arc length of an ellipse, whence the name derives. Jacobi's elliptic functions have found numerous applications in physics, and were used by Jacobi to prove some results in elementary number theory. A more complete study of elliptic functions was later undertaken by Karl Weierstrass, who found a simple elliptic function in terms of which all the others could be expressed. Besides their practical use in the evaluation of integrals and the explicit solution of certain differential equations, elliptic functions are at the crossroads of several branches of pure mathematics. The purpose of this note is to give a brief introduction to elliptic functions and related functions – theta functions. One of our exercises emphasizes their relation with field theory and Galois theory.

Section 1 recalls some basic theorems in complex analysis and definition of meromorphic functions. Section 2 gives the definition of elliptic functions and introduce the Weierstrass's construction. Section 3 gives the definition of theta functions and shows that how to construct elliptic functions using theta functions.

1. Meromorphic Functions

Let D be a **connected** open set in $\mathbb{C}$, a complex valued $f(z)$ defined on D is called an **analytic function** if $f'(z)$ exists everywhere in D.

Recall $f'(z)$ is the complex derivative defined by

$$\lim_{\delta\to 0}\frac{f(z+\delta)-f(z)}{\delta},$$

where δ goes to 0 at all the directions in $\mathbb{C}$.

Analytic functions have good properties that general smooth functions don't have.

Theorem 1.1. *If $F(z)$ is an analytic function on D, C is a simple counter-clockwise contour in D, then*

$$\int_C f(z)dz = 0.$$

For a point a in the domain enclosed by C,

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz = f(a).$$

Theorem 1.1 imply the the Theorems 1.2, 1.3, 1.4, 1.5 below.

Theorem 1.2. *If $f(z)$ is an analytic function on D, if $|f(z)|$ has a local maximal at some point in D, then $f(z)$ is a constant function.*

Theorem 1.3. *The derivative $f^{(n)}(z)$ of arbitrary order n exists, and*

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{1}{n!}f^{(n)}(a).$$

Theorem 1.4. *If $f(z)$ is an analytic function on D, for every $a \in D$, the Taylor expansion at a*

$$f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(z-a)^n+\dots$$

converges absolutely and uniformly on any closed disc $|z-a|\le r$ inside D.

Theorem 1.5. *If the zero points $\{a \mid f(a)=0\}$ has a limit point in D, then $f(z)=0$.*

A **meromorphic function** on D is a map $f : D \to \mathbb{C}\cup\{\infty\}$ such that
(1). If $f(a)=\infty$, then a is an isolated point in D, and there exists a

positve integer n, such that $\lim_{z\to a}(z-a)^n f(z)$ exists and is non-zero. (Such a is called the pole of $f(z)$, n is called the order of the pole).
(2). By (1), $D-f^{-1}(\infty)$ is an open set, $f(z)$ is analytic on $D-f^{-1}(\infty)$.

Example 1. A **rational function** is a meromorphic function on $\mathbb{C}$ of the form

$$f(z)=\frac{p(z)}{q(z)}$$

where $p(z)$ and $q(z)$ are polynomials, we may assume $p(z)$ and $q(z)$ have no common zeros. a is pole of $f(z)$ iff $q(a)=0$, its order is the multiplicity of a as a zero of $q(z)$.

Example 2. $f(z)=\frac{1}{e^z-1}$ is a meromorphic function on $\mathbb{C}$, whose poles are $2\pi i\mathbb{Z}$.

For a meromporphic function $f(z)$ on D, if $a\in D$ is a pole, $f(z)$ has a Laurent power series expansion at a

$$c_{-n}(z-a)^{-n}+\cdots+c_{-1}(z-a)^{-1}+\sum_{k=0}^{\infty}c_k(z-a)^k$$

where $c_{-n}\neq 0$.

Proposition 1.6. *The space of all analytic functions on D is an integral domain. The space of all meromorphic functions on D is a field.*

2. Elliptic Functions

Definition. An elliptic function is a function $f(z)$ meromorphic on $\mathbb{C}$ for which there exist two non-zero complex numbers ω_1 and ω_2 with $\frac{\omega_1}{\omega_2}\notin\mathbb{R}$, such that

$$f(z)=f(z+\omega_1),\quad f(z)=f(z+\omega_2)$$

for all $z\in\mathbb{C}$.

Denoting the "lattice of periods" by

$$\Lambda=\{m\omega_1+n\omega_2\mid m,n\in\mathbb{Z}\}.$$

It is clear that the condition

$$f(z)=f(z+\omega_1),\quad f(z)=f(z+\omega_2)$$

is equivalent to

$$f(z) = f(z + \omega)$$

for all $\omega \in \Lambda$.

We denote $\mathcal{M}(\Lambda)$ the space of all elliptic functions with lattice of periods Λ.

Proposition 2.1. *$\mathcal{M}(\Lambda)$ is a field.*

Just as a periodic function of a real variable is defined by its values on an interval, for example, a real variable periodic function function $f(x)$ with period a ($f(x+a) = f(x)$) is determined by its value on the interval $[0, a]$, an elliptic function with periods ω_1 and ω_2 is determined by its values on a fundamental parallelogram

$$\{u\omega_1 + t\omega_2 \mid 0 \le u, t \le 1\},$$

which then repeat in a lattice. If such a doubly periodic function $f(z)$ is analytic on whole $\mathbb{C}$, then $|f(z)|$ achieves on local maximum on the fundamental parallelogram, which is an absolute maximum by double periodicity, so $f(z)$ is a constant by Theorem 1.2. This proves

Theorem 2.1. *If an elliptic function $f(z)$ is analytic, then it is a constant function.*

With the definition of elliptic functions given above, the Weierstrass elliptic function $\wp(z)$ is constructed as follows: given a lattice Λ as above, put

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

To prove the convergence, we notice that on any compact disk defined by $|z| \le R$, and for any $|\omega| > 2R$, one has

$$\left| \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right| = \left| \frac{2\omega z - z^2}{\omega^2 (\omega - z)^2} \right| = \left| \frac{z\left(2 - \frac{z}{\omega}\right)}{\omega^3 \left(1 - \frac{z}{\omega}\right)^2} \right| \le \frac{10R}{|\omega|^3}$$

This implies that the series converges uniformly on $|z| \le R$, so we have a meromorphic function on $\mathbb{C}$ with poles on the lattice Λ.

The prove $\wp(z)$ has periods Λ, we notice that

$$\wp'(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$$

has periods Λ, so we have

$$\wp(z+\omega)-\wp(z)=C$$

is a constant, put $z=-\frac{\omega}{2}$, we see that $\wp(\frac{\omega}{2})-\wp(-\frac{\omega}{2})=C$, it is obvious that $\wp(z)$ is even function, so $C=0$. This proves $\wp(z)$ is an elliptic function with period Λ.

By writing $\wp$ as a Laurent series and explicitly comparing terms, one may verify that it satisfies the relation

$$\left(\wp'(z)\right)^2 = 4\left(\wp(z)\right)^3 - g_2\wp(z) - g_3 \tag{2.1}$$

where

$$g_2 = 60 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^4}$$

and

$$g_3 = 140 \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^6}.$$

This means that the pair $(\wp,\wp')$ parametrize an elliptic curve

$$y^2 = 4x^3 - g_2x - g_3.$$

Theorem 2.2. *The field $\mathcal{M}(\Lambda)$ is generated by $\wp(z)$ and $\wp'(z)$ over $\mathbb{C}$ subject to the relation (2.1)*

3. Theta Functions

The Jacobi theta function (named after Carl Gustav Jacobi) is a function defined for two complex variables z and τ, where z can be any complex number and τ is confined to the upper half-plane, which means it has positive imaginary part. It is given by the formula

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2\tau + 2\pi i n z) = 1 + 2\sum_{n=1}^{\infty} \left(e^{\pi i\tau}\right)^{n^2} \cos(2\pi n z)$$

If τ is fixed, this becomes a Fourier series for a periodic entire function of z with period 1; in this case, the theta function satisfies the identity

$$\vartheta(z+1;\tau) = \vartheta(z;\tau). \tag{3.1}$$

The function also behaves very regularly with respect to its quasi-period τ:

$$\vartheta(z+\tau;\tau) = \exp(-\pi i\tau - 2\pi i z)\,\vartheta(z;\tau). \tag{3.2}$$

(3.1) and (3.2) implies that for arbitrary integers a, b,

$$\vartheta(z+a+b\tau;\tau) = \exp(-\pi i b^2\tau - 2\pi i b z)\,\vartheta(z;\tau).$$

Theorem 3.1. *If $4n$ real numbers a_k, b_k, c_k, d_k $(k = 1, 2, \ldots, n)$ satisfy the conditions $\sum_{k=1}^n a_i - \sum_{k=1}^n c_i \in \mathbb{Z}$ and $\sum_{k=1}^n b_k = \sum_{k=1}^n d_k$, then*

$$\frac{\vartheta(z+a_1+b_1\tau;\tau)\,\vartheta(z+a_2+b_2\tau;\tau)\cdot\cdots\cdot\vartheta(z+a_n+b_n\tau;\tau)}{\vartheta(z+c_1+d_1\tau;\tau)\,\vartheta(z+c_2+d_2\tau;\tau)\cdot\cdots\cdot\vartheta(z+c_n+d_n\tau;\tau)}$$

is an elliptic function of period lattice $\mathbb{Z} + \mathbb{Z}\tau$.

Exercises.

Problem 1. This problem discusses an analog of Weierstrass' construction for trigonometric functions. Prove that for each integer $n \geq 2$, the series

$$f_n(z) = \sum_{k\in\mathbb{Z}} \frac{1}{(z-k)^n}$$

converges for $z \notin \mathbb{Z}$ and defines a periodic meromorphic function on $\mathbb{C}$ with period $\mathbb{Z}$. What is the relation of $f_n(z)$ with $\tan(\pi z)$?

Problem 2. Prove that

$$\vartheta(z;\tau) = \prod_{m=1}^{\infty}\left(1 - e^{2m\pi \mathrm{i}\tau}\right)\left[1 + e^{(2m-1)\pi\mathrm{i}\tau + 2\pi\mathrm{i}z}\right]\left[1 + e^{(2m-1)\pi\mathrm{i}\tau - 2\pi\mathrm{i}z}\right].$$

From this, prove that as function of z (with fixed τ), $\vartheta(z;\tau)$ has only simple zeros at $\frac{1}{2} + \frac{1}{2}\tau + m + n\tau$ $(m, n \in \mathbb{Z})$. Find all the poles and zeros of the elliptic function in Theorem 3.1.

Problem 3*. Suppose the lattice $\Lambda_1 \subset \Lambda_2$, prove that
(1). $\mathcal{M}(\Lambda_2)$ is a subfield of $\mathcal{M}(\Lambda_1)$.
(2). Prove that the field extension $\mathcal{M}(\Lambda_2) \subset \mathcal{M}(\Lambda_1)$ is a finite Galois

extension.
(3). Prove that the Galois group $G(\mathcal{M}(\Lambda_1)/\mathcal{M}(\Lambda_2))$ is isomorphic to the quotient group Λ_2/Λ_1.

Problem 4*. Write $\wp(z)$ for $\Lambda = \mathbb{Z}+\mathbb{Z}\tau$ as a product of theta functions as in Theorem 3.1. hint: you need Problem 2.

Problem 5*. Prove the converge of Theorem 3.1, i.e, every non-zero elliptic function of period lattice $\mathbb{Z}+\mathbb{Z}\tau$ can be written as a quotient of products of shifted theta functions as in Theorem 3.1.

References

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[2] William A. Schwalm, ectures on selected topics in mathematical physics : elliptic functions and elliptic integrals, Morgan & Claypool Publishers. 2015